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Abstract

We show how VEV's and condensates can be read off from the Schrödinger wave-functional without further calculation. This allows us to study non-perturbative physics by solving the Schrödinger equation. To illustrate the method we calculate fermion condensates from the exact solution of the Schwinger model, and other (1+1) dimensional models. The chiral condensate is seen to be a large-distance effect due to propagators reflecting off the space-time boundary.

1 Introduction

The non-perturbative calculation of expectation values is of central importance to QFT, as we learn from the study of everything from quark confinement and dynamical symmetry breaking to string/gauge theory duality. Instanton approximations, lattice gauge theory, and the study of supersymmetry all give us important insights into non-perturbative physics, but as yet, methods for continuum calculations in realistic models are extremely limited.

Another potential source of techniques for QFT is the functional Schrödinger equation. The wave-functional solutions of this equation give us a new perspective on insights generated by other methods, and provide some new calculational tools.

We still need to know how to read off interesting physics from the wave-functional. The calculation of expectation values from the wave-functional is something of a vexed issue, since taking the inner-product of an operator with respect to a QFT state formally involves functional integrations, and in a non-Gaussian theory, this requires a diagrammatic expansion and complicated calculations, even if the wave-functional is already known.

It turns out that this is misleading. In this paper, we will show how expectation values can be read off directly from the wave-functional, without further integrations. We will show how chiral symmetry breaking and confinement appear in this context in simple models.

In a previous paper [2], we solved the Schrödinger equation for the Schwinger model at zero and finite temperature. We will use this to illustrate our prescription for obtaining expectation values. Because the theory is non-Gaussian, exhibits many interesting phenomena such as a chiral condensate and phase transition, but remains exactly solvable, it provides a good illustration of general principles.

The non-zero value of the chiral condensate is a large-distance effect: the wave-functional is written in terms of correlation functions on a space-time boundary, while VEV's correspond to correlation functions in the bulk. As the boundary is taken to infinity, propagators reflecting off it can still give a non-zero correction to the perturbative VEV's. This correction is given by a large-distance limit of the boundary correlation functions.

This gives a mechanism whereby large-distance effects can be responsible for the dynamical generation of condensates. It would be interesting to see if other examples of dynamical symmetry breaking, such as confinement, could be understood in this way.

2 The Schrödinger wave-functional

The primary object of study in our investigations is the Schrödinger wave-functional, which can be interpreted as the density matrix between energy eigenstates at a finite temperature $T = 1/k\tau$. If we let $T \to 0$ then we get the vacuum wave-functional. We will represent the Schrödinger functional as a matrix element of the operator $e^{-\hat{H}\tau}$:

$$\Psi[\psi_1, \psi_2] = \langle \psi_1 | e^{-\hat{H}\tau} | \psi_2 \rangle. \tag{1}$$

This satisfies the time dependent Schrödinger equation $\hat{H}\Psi = -\tau\Psi$, and can be interpreted as the path integral of the exponentiated action over a Euclidean space-time region bounded by surfaces at $t = 0$ and $t = \tau$, on which the boundary conditions are specified by the arguments ψ_1 and ψ_2 respectively.

Imposing the boundary conditions corresponds to introducing sources on the boundary which diagonalize a maximal set of canonical field variables. The conjugate variables are represented by functional differentiation, and it is in such a representation that concrete calculations will be performed.

For bosons we can select Dirichlet or Neumann conditions, or in general some combination thereof. For fermions we found in [2] a set of local projection operators that can be used to specify the diagonal components. These boundary conditions have several advantages, for example:

1. They guarantee that Gauss' law is obeyed by the wave-functional.
2. When they are imposed the Dirac operator has no zero modes, which simplifies the quantization considerably.
3. They provide a set of equivalent representations for the wave-functional which may be used interchangeably.

Expectation values are obtained in the following manner: All operators have a well-defined action on the wave-functional, and their expectation value is the functional trace of the resulting object. From the path-integral point of view this corresponds to operator insertions on the boundary, followed by a functional integration that glues the two boundaries together. This is equivalent to the usual prescription which inserts the operators on a cylinder of radius τ. If we started with the vacuum functional, the procedure is much the same; we glue together two copies of the half-plane with suitable boundary insertions, and integrate over boundary values. The boundaries disappear, and full Euclidean invariance is restored.

Up to certain simple rescalings the wave-functional can itself be interpreted as a generating functional of expectation values or equal-time correlation functions. Then the (usually non-Gaussian) functional integrations described in the last paragraph are unnecessary, and most of the physically interesting information can be read off from the wave-functional without too much further work.

For simplicity we will consider in this paper mostly cases where the renormalization of the Schrödinger representation requires no additional counterterms on the boundary, apart from the usual ones. In gauge theories (with or without fermions) this is guaranteed by gauge invariance.

We will need to take account of boundary effects, ie. relate correlation functions on the boundary to those deep in the bulk. This will enable us to account for large-distance effects which are responsible for the dynamical generation of condensates that are responsible for chiral symmetry breaking. We speculate that a similar phenomenon might be involved in the dynamical generation of condensates leading to confinement.

3 Calculation of correlation functions

The vacuum functional can be written in the form:

$$\Psi[\varphi] = exp(-W[\varphi]) = \int D\phi \exp(-S + S_B) \tag{2}$$

where S is the euclidean action, and S_B is a boundary term imposing the appropriate boundary conditions. These boundary conditions can be changed by performing a functional Fourier transform—for example, if we started with the "field" representation where ϕ is diagonal, then

$$\tilde{\Psi}[\tilde{\varphi}] = \int D\varphi \Psi[\varphi] e^{i\varphi\tilde{\varphi}} \tag{3}$$

is the corresponding vacuum functional in the "momentum" representation where the momenta conjugate to ϕ are diagonal.

Now according to the prescription for obtaining VEV's, we can write

$$\exp(-F[J]) = \int D\varphi \; |\Psi[\varphi]|^2 \; e^{iJ\varphi}, \tag{4}$$

where $F[J]$ is the generating functional for connected equal-time correlation functions, so that the vacuum expectation of some function $G[\phi]$ is given by

$$G[\frac{\delta}{\delta J}] \exp(-F[J])|_{J=0}. \tag{5}$$

Now for a wide range of interesting cases, the wave functional can be shown to be real, in which case

$$|\Psi[\varphi]|^2 = \exp(-2W[\varphi]) \tag{6}$$

Compare the two objects:

$$\begin{aligned} \exp(-F[J]) &= \int D\varphi \exp(-2W[\varphi]) e^{iJ\varphi} \\ \exp(-\tilde{W}[\tilde{\varphi}]) &= \int D\varphi \exp(-W[\varphi]) e^{i\tilde{\varphi}\varphi} \end{aligned} \tag{7}$$

If we can find a scaling of fields, coupling constants and other parameters that effects the transformation $W[\varphi] \to 2W[\varphi]$, then we can essentially identify the wave-functional in the second line with the generating functional in the first.

Let us see how this works in practice. For a free theory, $W[\varphi]$ is quadratic, and (up to an overall constant) we can achieve the desired effect by rescaling the argument: $\varphi \to \sqrt{2}\varphi$. Then

$$F[J] = \tilde{W}[J/\sqrt{2}] \tag{8}$$

More generally, this rescaling will give the correct quadratic term for *any* theory, so that the correct two-point correlators can easily be extracted from the appropriate vacuum functional. This technique is similar to the "plasma analogy" that is of importance in the study of the fractional quantum Hall effect [6] and other phenomena in statistical physics.

We can generalize this to non-free theories as follows. Consider a theory with a mass gap. For slowly varying sources, the vacuum can be expanded in integrals over space of local functions of the fields and their derivatives at a single space-time point.[1]

Since $W[\varphi]$ is dimensionless, the integrand has dimension $(length)^{-d}$, where d is the dimension of space. We can rescale all fields and dimensionful parameters according to their dimension in such a way that $W[\varphi] \to 2W[\varphi]$. Note that we will also require derivatives to scale, which means scaling the arguments of the fields in an appropriate way. The required prescription is to ensure that objects of dimension $(length)^{\alpha}$ scale as $2^{-\alpha/d}$. These scalings translate into a well-defined scaling prescription for $\tilde{W}[J]$ that makes it equal to $F[J]$.

The existence of a local expansion is not really necessary for this procedure to work; it is enough that the vacuum functional is derived from a local action. The point is that an appropriate rescaling of dimensionful parameters is enough to account for the difference between the vacuum functional and the generating functional of equal time correlation functions.

4 Large distance corrections to bulk correlation functions

To illustrate how large distance effects can modify correlation functions in the bulk, consider a particle propagating between two points A and B in the interior of some bounded Euclidean region $\tau \leq t \leq \infty$. The propagator contains contributions from the straight path from A to B, as well as the path that reflects off the boundary at $t = \tau$.

Now suppose that we restore Euclidean invariance by taking $\tau \to -\infty$. Usually the latter contribution to the propagator is suppressed, eg. for a free particle, where the action is proportional to the path length. An exception to this is the chiral condensate, which is non-zero only for an odd number of reflections.

How does this affect the calculation of expectation values? Suppose we have a theory containing complex fermions ψ and $\psi^{\dagger}$. On the boundary $t = \tau$ we will impose the boundary conditions discussed in [2], diagonalising $Q_+\psi$ and $\psi^{\dagger}Q_-$, where $Q_{\pm} = \frac{1}{2}(1 \pm Q)$ where $Q = a\gamma^0 \pm b\gamma^1 + c\gamma^5$ and $a^2 + b^2 + c^2 = 1$.

Consider the chiral condensate $< \bar{\psi}(x,0)\psi(x,0) >$. We can rewrite this as

$$< \bar{\psi}(x,0)Q_+\psi(x,0) > + < \bar{\psi}(x,0)Q_-\psi(x,0) > . \tag{9}$$

These expressions are to be interpreted as propagators between coinciding points in the interior of the bounded Euclidean region, and as $\tau \to \infty$ they should tend to the vacuum expectation values of the corresponding fermion currents.

[1] This is a simplification. It may be necessary to rewrite the vacuum functional in terms of other variables [7], or even to express it as a limit of some regulated object [2] in order to achieve this result. But some sort of local expansion can generally be found.

Now consider the first term in (9). In the cases we will consider, the dominant contribution to this term comes from the shortest path via the boundary at $t = \tau$. We will choose $Q = \gamma^5$, so that $\bar{\psi} Q_+ = -\psi^\dagger Q_- \gamma^0$. According to the method of images, and since $\psi^\dagger Q_-$ and $Q_+ \psi$ are freely integrated over on the boundary, we have

$$< \bar{\psi}(x,0) Q_+ \psi(x,0) > = < \psi^\dagger(x + 2\tau, \tau) Q_- \gamma^0 Q_+ \psi(x,\tau) > . \qquad (10)$$

The second term in (9) is identical, but with $Q = -\gamma^5$, so that $Q_\pm \to Q_\mp$. As we take the limit $\tau \to \infty$, we see that the chiral condensate is given by the large-distance limit of the boundary propagators.

Having re-expressed the chiral condensate in terms of boundary correlation functions of non-diagonal fields, we can now use the results of the last section to extract the value of the chiral condensate from the vacuum functional. In [2] we found the vacuum functional for the Schwinger model at zero and finite temperature. Here we quote the result, generalising it for amusement to the case of multiple fermion flavours.

The representation is:[2]

$$Q_+ \hat{\psi} \sim u, \qquad \hat{\psi}^\dagger Q_- \sim \tilde{u} \qquad (11)$$

$$Q_- \hat{\psi} \sim \frac{\delta}{\delta \tilde{u}}, \qquad \hat{\psi}^\dagger Q_+ \sim \frac{\delta}{\delta u} \qquad (12)$$

and the vacuum wave-functional is

$$\begin{aligned} \Psi[u, \tilde{u}] &= \sum_{a=0}^{\infty} \frac{1}{a!} \prod_{n=0}^{a} \left[\frac{2}{\pi} \int dx_n dy_n \tilde{u}(x_n) \gamma^0 u(y_n) \mathcal{P} \frac{1}{x_n - y_n} \right. \\ & \left. \times \exp \left\{ \sum_{i,j=1}^{a} \Phi(x_i - y_i) - \sum_{j>i=1}^{a} [\Phi(x_i - x_j) + \Phi(y_i - y_j)] \right\} \right], \end{aligned} \qquad (13)$$

where

$$\Phi(x) = \frac{1}{N_f} \int \frac{dp}{2\pi} \left(\frac{1}{|p|} - \frac{\sqrt{p^2 + m^2}}{p^2} \right) (1 - \cos(px)) \qquad (14)$$

for the Schwinger model at zero temperature, and

$$\Phi(x) = \frac{1}{N_f} \int \frac{dp}{2\pi} \left(\coth(p\tau) \frac{1}{p} - \coth(\sqrt{p^2 + m^2}\tau) \frac{\sqrt{p^2 + m^2}}{p^2} \right) (1 - \cos(px)) \qquad (15)$$

for the Schwinger model at temperature $T = 1/k\tau$, where $m = \sqrt{\frac{N_f e^2}{\pi}}$ is the Schwinger mass for N_f flavours.

According to our scaling prescription, we can calculate the boundary propagator in (10) as

[2] The source for the electromagnetic field can be gauged away completely, and the vacuum is expressed as a function of fermion fields only.

$$
\begin{aligned}
<\psi^\dagger Q_-\gamma^0 Q_+\psi> &= \lim_{x-y\to\infty} \frac{\delta}{\delta u(x)}\gamma^0 \frac{\delta}{\delta \tilde{u}(y)} \Psi[\frac{1}{\sqrt{2}}u, \frac{1}{\sqrt{2}}\tilde{u}]|_{u,\tilde{u}=0} \\
&= N_f \mathrm{tr}(Q_-\gamma^0 Q_+\gamma^0) \lim_{x\to\infty} \frac{1}{\pi x} \exp(\Phi(x)). \qquad (16)
\end{aligned}
$$

$\Phi(x)$ has the asymptotic form

$$
\Phi(x) \sim \frac{1}{N_f}(\gamma + \ln(mx/2)) + O(e^{-mx}) \qquad (17)
$$

for both zero and finite temperature, and $\mathrm{tr}(Q_\mp\gamma^0 Q_\pm\gamma^0) = 1/2$. Putting this all together, we have

$$
<\bar{\psi}\psi>= \frac{me^\gamma}{2\pi} \qquad (18)
$$

for $N_f = 1$, and $<\bar{\psi}\psi>= 0$ for $N_f > 1$.

Because of the anomaly, the vacuum angle can be altered by a chiral rotation of the fermion fields:

$$
\hat{\psi} \to e^{i\theta\gamma^5/2}\hat{\psi}, \qquad \hat{\psi}^\dagger \to \hat{\psi}^\dagger e^{-i\theta\gamma^5/2} \qquad (19)
$$

Under this transformation we have for $N_f = 1$

$$
\begin{aligned}
<\bar{\psi}\psi> &\to <\bar{\psi}e^{i\theta\gamma^5}\psi> \\
&= e^{i\theta}<\bar{\psi}Q_+\psi> + e^{-i\theta}<\bar{\psi}Q_-\psi> \\
&= \frac{me^\gamma}{2\pi}\cos\theta \qquad (20)
\end{aligned}
$$

The Schwinger model in a finite box exhibits a second order phase transition at $T = 0$, which has been studied numerically [3]. The above analysis is readily extended to this case.

Consider a more general model with a four-fermi interaction

$$
\frac{1}{2}\int dx \int dy \psi^\dagger(x)\psi(x)u(x-y)\psi^\dagger(y)\psi(y) \qquad (21)
$$

The Schwinger model corresponds to $u(x) = -e^2|x-y|$, and in general the vacuum wave-functional is given by (13) with

$$
\Phi(x) = \int \frac{dp}{2\pi}\frac{1}{|p|}\left(1 - \frac{\tilde{u}(p)}{\pi}\right)^{\frac{1}{2}}(1-\cos(px)) \qquad (22)
$$

with $\tilde{u}(p)$ the Fourier transform of $u(x)$. To get a non-zero chiral condensate, the large-distance behaviour of $\Phi(x)$ must have the same leading-order logarithmic behaviour as for the Schwinger model. The massive Schwinger model is an example. Note that the actual value of the condensate is determined by the small-distance behaviour of $\Phi(x)$ and may in fact be extracted from the vacuum energy using mass perturbation theory [5].

5 Discussion

The disadvantage of our prescription for obtaining VEV's is that it only works for fields that are non-diagonal in our chosen representation for the vacuum functional. Thus we cannot straightforwardly obtain VEV's that are mixed functions of fields and their canonical conjugates, such as the gluon condensate. Also, we are in general tied to a specific representation.

For example, the vacuum wave-functional for (2+1) dimensional Yang-Mills is given in [7] for the A-representation (A_μ diagonal), but the corresponding calculation in the electric representation is less straightforward (though two-point correlators are easily obtained by restricting to the abelian part of the wave-functional.

Nevertheless, it would in principle be possible to calculate expectation values of physically relevant operators such as the Wilson loop from the appropriate wave-functional, if it could be found. It is likely that large-distance effects, such as that responsible for chiral condensates in the simple models we have considered, would need to be taken into account in such calculations.

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